On accuracy of an asymptotic formula for the phase of arrival at a resonance in slow-fast Hamiltonian systems

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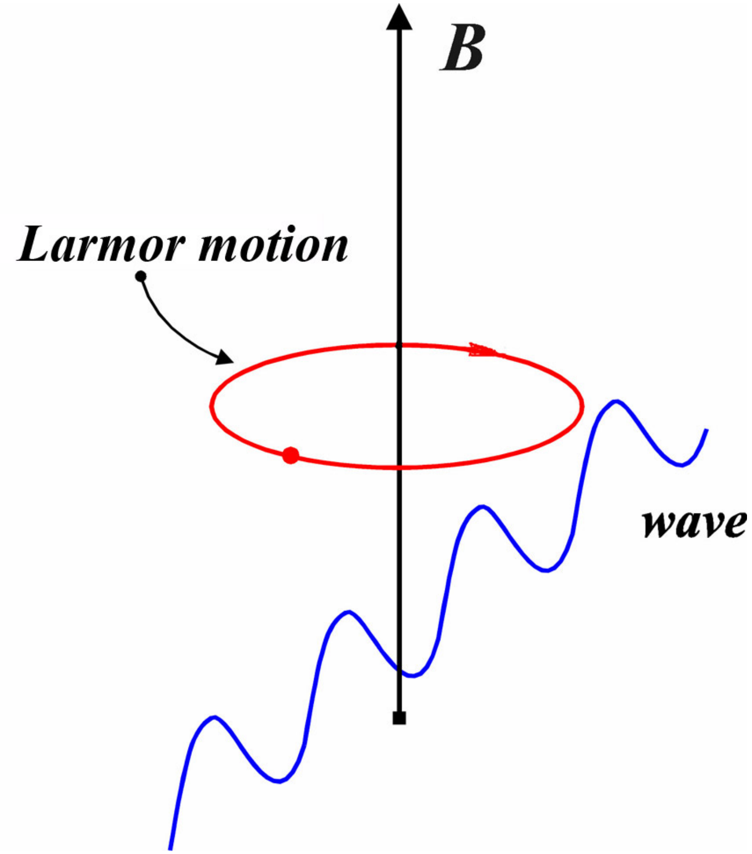
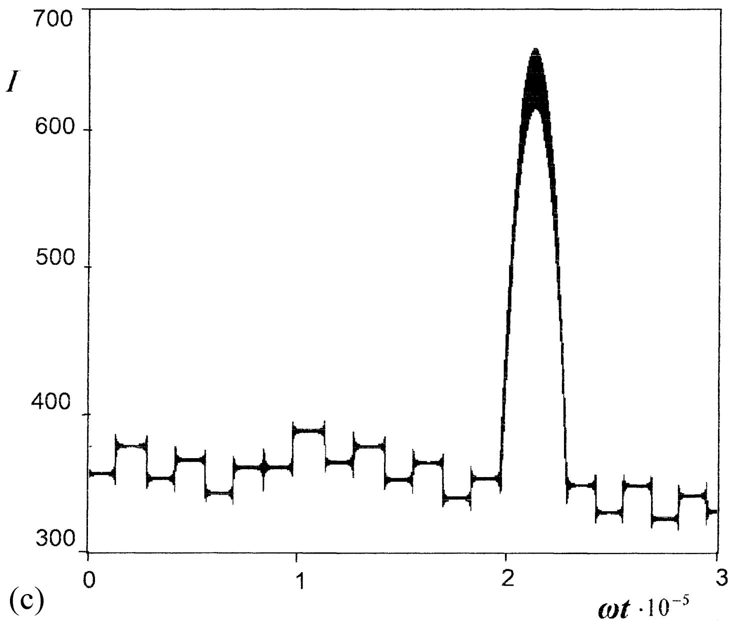
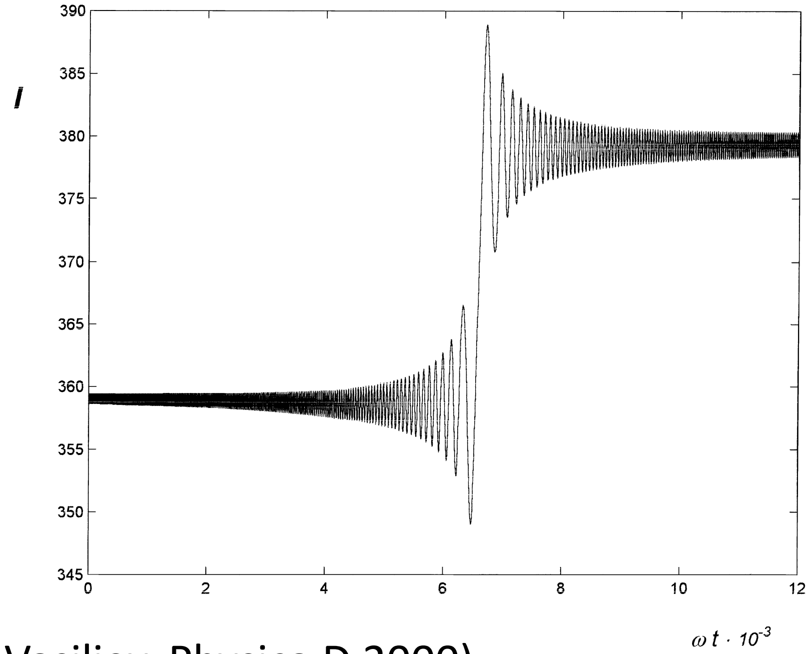
Motivation: motion of a charged particle in a background magnetic field and a high-frequency wave exited in a plasma







From: A.Itin, N., A.Vasiliev, Physica D,2000)

A model problem: one-frequency slow-fast Hamiltonian system

Canonically conjugate variables: $(I, \varphi),\ (y, \varepsilon^{-1}x)$, $I \in \mathbb{R}^1, \varphi \in \mathbb{S}^1$

$$0 < \varepsilon \ll 1$$

Hamilton's function: $H(I, \varphi, y, x, \varepsilon) = H_0(I, y, x) + \varepsilon H_1(I, \varphi, y, x, \varepsilon)$

Hamilton's equations:

$$\dot{I} = -\varepsilon \frac{\partial H_1}{\partial \varphi}, \quad \dot{\varphi} = \frac{\partial H_0}{\partial I} + \varepsilon \frac{\partial H_1}{\partial I},$$

$$\dot{y} = -\varepsilon \frac{\partial H_0}{\partial x} - \varepsilon^2 \frac{\partial H_1}{\partial x}, \quad \dot{x} = \varepsilon \frac{\partial H_0}{\partial y} + \varepsilon^2 \frac{\partial H_1}{\partial y}$$

Thus, I, y, x are *slow variables*, φ is the *fast phase*.

The averaged system (*an adiabatic approximation*):

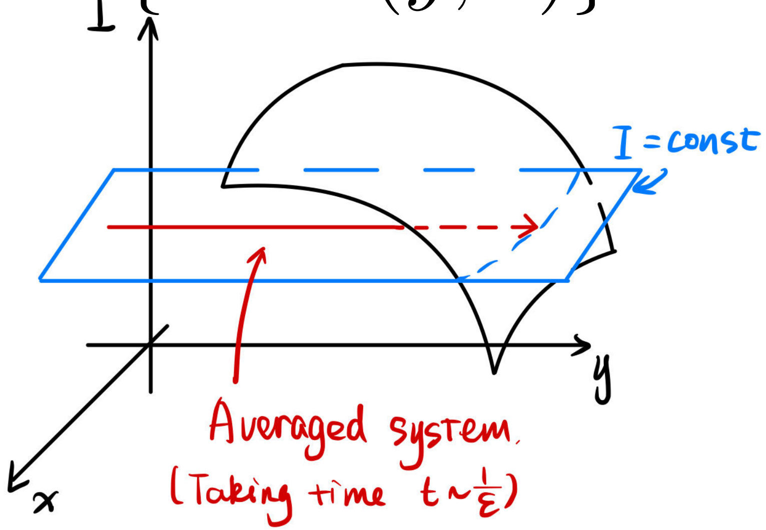
$$\dot{I} = 0, \quad \dot{y} = -\varepsilon \frac{\partial H_0(I, y, x)}{\partial x}, \quad \dot{x} = \varepsilon \frac{\partial H_0(I, y, x)}{\partial y}.$$

The (unperturbed) frequency of the fast phase φ is

$$\omega_0(I, y, x) = \frac{\partial H_0(I, y, x)}{\partial I}$$

The frequency vanishes on *a resonant surface* $\{I = a(y, x)\}$.

We assume that the trajectory of the averaged system transversally crosses the resonant surface.



Dynamics near the resonant surface

Introduce the normalised distance from the resonant surface
$P=(I-a(x,y))/\sqrt{\varepsilon}+O(\sqrt{\varepsilon})$ and rescale time to $\theta=\sqrt{\varepsilon}t$.

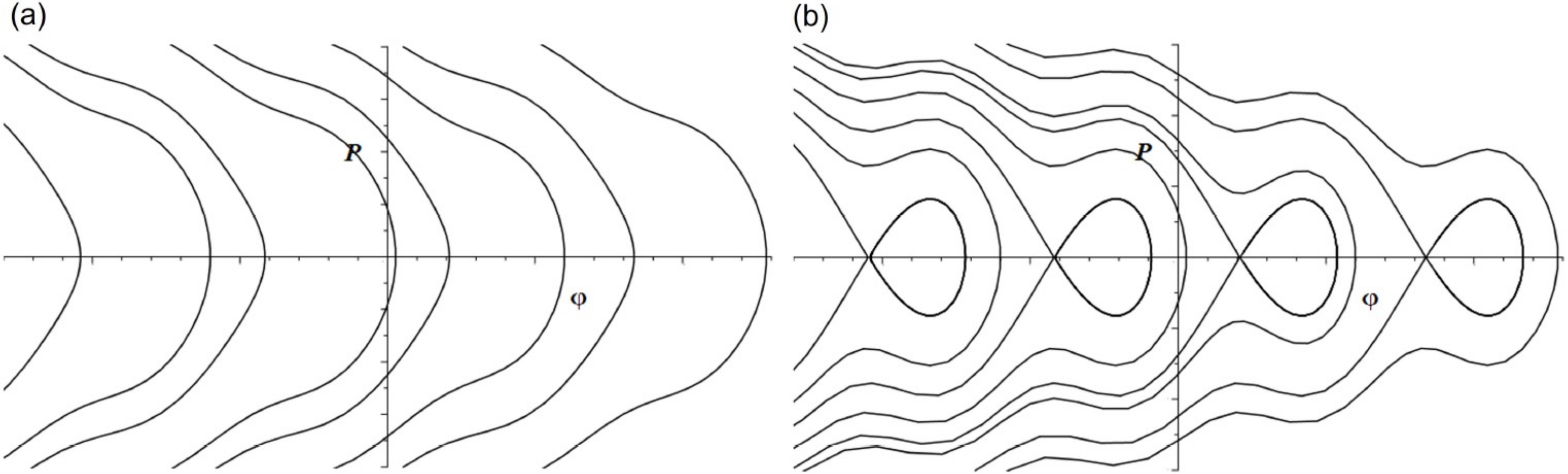
Expand Hamiltonian in $O(\sqrt{\varepsilon})-$ neighbourhood of the resonant surface:

- dynamics of y, x is approximately described by the Hamiltonian
  $$\sqrt{\varepsilon}H_0(a(y,x),y,x)$$
- dynamics of P,φ is approximately described by the Hamiltonian

$$\mathcal{E}(P,\varphi,y,x)=\frac{1}{2}\alpha(y,x)P^2+H_1(a(y,x),\varphi,y,x,0)+b(y,x)\varphi$$

that depends on y, x .

Phase portraits of P, φ for frozen y, x .



Pseudophase: $$\Xi(\varphi; y, x) = \frac{1}{2\pi}\left(\varphi + \frac{\tilde{H}_1(a(y,x), \varphi, y, x)}{b(y,x)}\right)$$

Pseudophase is related to value of Hamiltonian $\mathcal{E}$ at $P = 0$ (i.e. on the resonant surface); $\tilde{H}_1$ is the purely periodic part of H_1 at $\varepsilon = 0$.

Improved adiabatic approximation

Canonically conjugate variables: $(J, \psi),\ (\eta, \varepsilon^{-1}\xi), \quad J \in \mathbb{R}^1, \psi \in \mathbb{S}^1$

Hamilton's function: $\mathcal{H}(J, \eta, \xi, \varepsilon) = H_0(J, \eta, \xi) + \varepsilon \bar{H}_1(J, \eta, \xi)$

Hamilton's equations

$$\dot{J} = 0, \quad \dot{\psi} = \frac{\partial H_0}{\partial J} + \varepsilon \frac{\partial \bar{H}_1}{\partial J}$$

$$\dot{\eta} = -\varepsilon \frac{\partial H_0}{\partial \xi} - \varepsilon^2 \frac{\partial \bar{H}_1}{\partial \xi}, \quad \dot{\xi} = \varepsilon \frac{\partial H_0}{\partial \eta} + \varepsilon^2 \frac{\partial \bar{H}_1}{\partial \eta}$$

Here $\bar{H}_1$ is the average of H_1 over φ at $\varepsilon = 0$.

Initial conditions (for t=0).

For the exact system:

$$(I_0, \varphi_0, y_0, x_0)$$

For improved adiabatic approximation:

$$(J_0, \psi_0, \eta_0, \xi_0) = (I_0, \varphi_0, y_0, x_0) + O(\varepsilon)$$

Additional notation

Consider solution of the perturbed system with initial data (I_0, φ_0, y, x_0).

Denote, for *slow time* $\tau = \varepsilon t$:

$I_0, \eta(\tau), \xi(\tau)$ - solution in adiabatic approximation,

τ_* - slow time of its arrival at resonance, $\omega_0 \left(I_0, \eta(\tau), \xi(\tau)\right) = 0$

$$y_* = \eta(\tau_*), x_* = \xi(\tau_*)$$

$J_0, \eta_a(\tau), \xi_a(\tau)$- solution in improved adiabatic approximation,

$\tau_{*,a}$ - slow time of its arrival at resonance, $\omega_0 \left(J_0, \eta_a(\tau), \xi_a(\tau)\right) = 0$

$\omega_1(I, y, x) = \partial \bar{H}_1(I, y, x)/\partial I$ - frequency adjustment.

Formula for (pseudo)phase at arrival to resonance

For simplicity of formulations, assume that there are no equilibria at the phase portraits of the system near the resonance.

Denote:

φ_e - value of the phase at arrival at the resonance

$\Xi_e = \Xi(\varphi_e; y_*, x_*)$ - the corresponding of pseudophase

$$2\pi\Xi_e = \varphi_0 + \frac{1}{\varepsilon}\int\limits_0^{\tau_{*,a}} (\omega_0\left(J_0, \eta_a(\tau), \xi_a(\tau)\right) + \varepsilon\omega_1\left(J_0, \eta_a(\tau), \xi_a(\tau)\right) d\tau + O(\sqrt{\varepsilon})$$

The estimate of the error is optimal.

Recall that $$\Xi(\varphi; y, x) = \frac{1}{2\pi}\left(\varphi + \frac{\tilde{H}_1(a(y,x), \varphi, y, x)}{b(y,x)}\right)$$

Analytical example.

Consider a Hamiltonian that depends on the slow time:

$$H = \frac{1}{2}(I-\tau)^2 + \varepsilon u(\tau)\sin\varphi, \qquad \tau = \varepsilon t$$

where $u(\tau)$ is a smooth function, u(0) = 0. Equations of motion are

$$\dot I = -\varepsilon u(\tau)\cos\varphi, \;\; \dot\varphi = I - \tau$$

In this example, $\tau_* = \tau_{*,a} = I_0$. One can show that

$$2\pi(\Xi_e - \Xi_{teor}) = \sqrt{\varepsilon}u'(\tau_*)K + O(\varepsilon),$$

$$K = \int_{-\infty}^{\varphi_a} \frac{\sin\varphi + u(\tau_*)\sin\varphi_a\cos\varphi}{\sqrt{2\left(u(\tau_*)\sin\varphi_a + \varphi_a - u(\tau_*)\sin\varphi - \varphi\right)}}d\varphi$$

where φ_a is a root of equation $I_0^2/(2\varepsilon) + \varphi_0 = u(\tau_*)\sin\varphi_a + \varphi_a$.

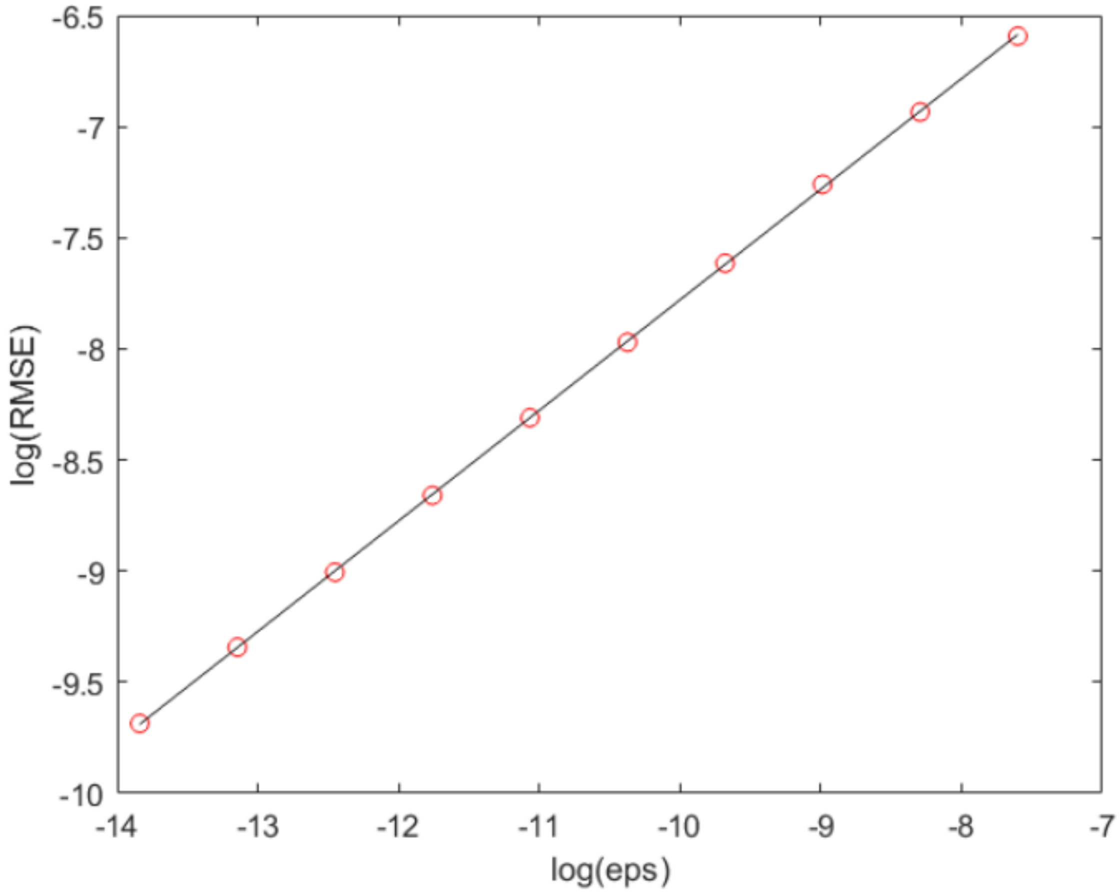
Value K is not identically 0. Thus, the estimate $O(\sqrt{\varepsilon})$ of the error term in the asymptotic formula for the pseudophase is optimal.

Numerical test

Consider a Hamiltonian that depends on the slow time:

$$H = \frac{1}{2}(I - \tau^2)^2 + \varepsilon(\frac{1}{2} + I)\sin\varphi\,.$$

A numerical check of accuracy of the formula for pseudophase at the resonance gives the slope of the fit function 0.4977.



Scheme of the proof

Consider the function:

$$E = \frac{1}{2\varepsilon}\alpha(y,x)(I - a(y,x))^2 + H_1(I,\varphi,y,x,\varepsilon) + b_{*,a}\varphi$$

This is an approximate first integral of motion near the resonant surface. Here $b_{*,a} = b(\eta(\tau_{*,a}),\xi(\tau_{*,a}))$. Calculate the time derivative of this function along solution and take the integral of this derivative from 0 to the time t_e of arrival into resonance. Compare two expressions of this integral. On the one hand

$$\int_0^{t_e} \dot{E} dt = E|_{t=t_e} - E|_{t=0}$$

$$= H_1(I_e,\varphi_e,y_e,x_e,\varepsilon) + b_{*,a}\varphi_e - \frac{1}{2\varepsilon}\alpha(y_0,x_0)(I_0 - a(y_0,x_0))^2$$

$$- H_1(I_0,\varphi_0,y_0,x_0,\varepsilon) - b_{*,a}\varphi_0.$$

On the other hand, we should integrate from 0 to t_e the expression

$$\dot{E} = \frac{1}{\varepsilon}\alpha(y,x)(I - a(y,x))(\dot{I} - \varepsilon a'(y,x)) + \frac{1}{2}\alpha'(y,x)(I - a(y,x))^2 + \frac{\partial H_1}{\partial I}\dot{I} + \frac{\partial H_1}{\partial \varphi}\dot{\varphi} + \frac{\partial H_1}{\partial y}\dot{y} + \frac{\partial H_1}{\partial x}\dot{x} + b_{*,a}\dot{\varphi}$$

Prime denotes derivative with respect to τ.

Many terms are cancelled near the resonance in the principal approximation. This simplifies estimates in a perturbation theory. Comparison of two expressions for the integral leads to the final result.

References

1. Artemyev, A., Neishtadt, A. and Vasiliev, A., Mapping for Nonlinear Electron Interaction with Whistler-Mode Waves, *Phys. Plasmas*, 2020, vol. 27, p.042902.

2. Gao, Y., Neishtadt, A. and Okunev, A., On Phase at a Resonance in Slow-fast Hamiltonian Systems, Regular and Chaotic Dynamics, 2023, vol. 28, pp 581--608.

Спасибо!