HAGEN-POISEUILLE FLOW LINEAR INSTABILITY

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Abstract. In the linear theory of hydrodynamic stability of the Hagen–Poiseuille (HP) flow in a round pipe, based on periodic disturbances, it is stable for any finite threshold Reynolds number Re_{th} that contradicts experimental data. Such disturbances do not correspond to the observation data where only quasi periodic disturbances fields (R. J. Leite, JFM 1959; D. D. Joseph, N. Y. 1976) are considered. We suggest here linear theory based on quasi periodic disturbances. We use energy and Galerkin's approximation methods taking into account existence of different periods for different radial modes corresponding to the equation of evolution of extremely small axially symmetric velocity field tangential component disturbances. Obtained for the HP flow linear instability realization minimal value $Re_{th}(p)=448$ (when $p=1.527$) agrees with the threshold value $\mathrm{Re}_{th} = 420$ for Tolmin-Shlihting waves arising in the boundary layer.

1. Introduction

The problem of understanding of the turbulence arising mechanism for the Hagen-Poiseuille (HP)[1] flow exists more than century because of the linear theory result of exponential stability of the HP flow with respect to disturbances with extremely small amplitude for arbitrary large Reynolds numbers [1-4]. It's contradiction with experiments is explained by an assumption of permissibility of the HP flow instability with respect to disturbances of sufficiently large finite amplitude [5-10]. At the same time, O. Reynolds noted [1]: high sensitivity of Re_{th} may depend on the space-time wave characteristics of the disturbances (see [11]). Here, we introduce an additional to the Reynolds number parameter p characterizing frequency-wave features

[1] HP flow is a laminar stationary flow of the uniform viscous incompressible fluid along the static straight linear and unbounded in length pipe with round, same along the whole pipe axis, cross section

of the disturbances (similar to ones of [12-16]), and get a finite minimal threshold Reynolds number $Re_{th}=448$ for $p=1.53$ of linear exponential instability. Obtained threshold Reynolds number for linear exponential instability for the HP flow is close to $Re_{th}=420$ for Tolmin-Shlihting (TS) waves arising due to the near boundary action of the molecular viscosity [17-19]; also, instability regions bounded by the curves of neutral stability are similar. This confirms expected similarity of viscous dissipative realization mechanisms of instability for the HP flow and for TS waves excitation. In Section 2, the problem under consideration is stated. In Section 3, energy approach and Galerkin's method respectively are used to get threshold Reynolds number for HP flow linear instability. In Section 4, obtained results are compared versus known data, and discussions and conclusions are given.

2. Statement of the problem

Let us consider representation [4] of the HP flow in the cylindrical reference frame (z,r,φ):

$$V_{0r} = V_{0\varphi} = 0, V_{0z} = V_{\max}(1 - r^2/R^2), \qquad V_{\max} = \frac{R^2}{4\rho\nu}\frac{\partial p_0}{\partial z} \text{ w,}$$

where the fluid density $\rho = const$, $\partial p_0 / \partial z$ is the constant pressure gradient, R is pipe radius, and ν is the coefficient of kinematic fluid viscosity. If $\partial V_\varphi / \partial \varphi = 0$, linear instability of the HP flow can be defined only by V_φ meeting (1), where y=r/R, x=z/R, $\tau = t\nu / R^2; \mathrm{Re} = V_{\max} R/\nu$.

$$\frac{\partial V_\varphi}{\partial \tau} + \mathrm{Re}(1-y^2)\frac{\partial V_\varphi}{\partial x} = \Delta V_\varphi - \frac{V_\varphi}{y^2}; \Delta = \frac{1}{y}\frac{\partial}{\partial y} y \frac{\partial}{\partial y} + \frac{\partial^2}{\partial x^2}; \quad , \qquad (1)$$
$$V_\varphi(y=1) = 0; V_\varphi(y=0) = 0$$

Seek solution of (1) as $V_\varphi = e^{\lambda\tau} V; V = \sum_{n=1}^{N} A_n(x) J_1(j_{1,n} y); J_1(j_{1,n}) = 0;$

$\lambda = \lambda_1 + i\lambda_2 \; V = V_1 + iV_2, V^* = V_1 - iV_2;\; A_n(x) = A_n(x + T_n);$

$T_n = 1/j_{1,n}; \max T_n = 1/j_{1,1}; i^2 = -1 \quad J_1$ - is the Bessel function of the first order and the value N must be considered as infinite to obtain the exact solution of (1).

3. Energy consideration and galerkin's method

Let's consider evolution of average energy (on the unit of mass):

$$E = \left\langle V_\varphi V_\varphi^* \right\rangle / 2 = e^{2\lambda_1 \tau} \left\langle VV^* \right\rangle / 2; I_0 = \left\langle VV^* \right\rangle = 2\int_0^1 dyy \frac{1}{T_{\max}} \int_0^{T_{\max}} dx VV^*$$

From (1) it is possible to obtain an equation for λ_1. When $A_n(x) = A_{0n} \exp(i2\pi\alpha_n x + i2\pi\beta_n)$, we may obtain the following criterion of the HP flow linear instability (only when c>0):

$$\mathrm{Re} > \mathrm{Re}_{Eth} = I_2 / I_1 = \frac{a^2 + \alpha_1^2 b^2}{\alpha_1 c}; I_2 / 2A_0^2 = a^2 + \alpha_1^2 b^2;$$

$$I_1 / 2A_0^2 = \alpha_1 c; c = c_0(p) + c_1; c_0(p) =$$

$$\frac{2}{j_{1,2}^{1+k}} \left(\frac{|q_{12}| \sin^2(\pi p)}{j_{1,1}^{1+k}} + \sum_{n=3}^{N} \frac{|q_{2n}| (-1)^{n+1} \sin^2(\pi(p - p_n))}{j_{1,n}^{1+k}} \right);$$

$$c_1 = \sum_{n=3}^{N} \left[\frac{2|q_{1n}|(-1)^n \sin^2(\pi p_n)}{j_{1,1}^{1+k} j_{1,n}^{1+k}} - \sum_{\substack{m=3 \\ m \neq n}}^{N} \frac{|q_{nm}|(-1)^{n+m} \sin^2(\pi(p_n - p_m))}{j_{1,n}^{1+k} j_{1,m}^{1+k}} \right];$$

$$a = \left[\sum_{n=1}^{N} \frac{J_2^2(j_{1,n})}{j_{1,n}^{2k}} \right]^{1/2}; b = 2\pi \left[\frac{J_2^2(j_{1,1})}{j_{1,1}^{2+2k}} + p^2 \frac{J_2^2(j_{1,2})}{j_{1,2}^{2+2k}} + \sum_{n=3}^{N} \frac{p_n^2 J_2^2(j_{1,n})}{j_{1,n}^{2+2k}} \right]^{1/2}$$

$$q_{nm} = q_{mn} = 2\int_0^1 dyy^3 J_1(j_{1,n} y) J_1(j_{1,m} y); q_{nm} > 0, n + m = 2k$$

$q_{nm} < 0, n + m = 2k + 1, k = 1,2,...$ Minimization of Re_{Eth} on the value of α_1 gives the following representation for criterion of linear instability of the HP flow ($\alpha_1 = \alpha_{1\min} = a / b$, $p_n = \alpha_n / \alpha_1 = j_{1,n} / j_{1,1}$):

$\mathrm{Re} > \mathrm{Re}_{\substack{Eth\min \\ \alpha_1 = \alpha_{1\min}}} = 2ab / c$ We may consider the more special representation with only one free independent parameter $p_2 = p$ on which we may minimize $\mathrm{Re}_{\substack{Eth\min \\ \alpha_1 = \alpha_{1\min}}}(p)$. Dependence of this function on p is presented in Fig.1,b. Thus, only when $\nu \neq 0$ in (1), it is possible to expect realization of the HP flow viscous dissipative instability for

Reynolds numbers $Re > \mathrm{Re}_{th}$. Instability of HP flow obviously is not realizable in the case of the pure periodic variability of V_φ along the tube (see also (24.7) in [2]) when $I_1 = 0$ and $\lambda_1 < 0$.

With Galerkin–Kantorovich method, for the coefficients A_n, from (1), we get the following dimensionless system of equations:

$$(\lambda + j_{1,m}^2)A_m - \frac{\partial^2 A_m}{\partial x^2} + \mathrm{Re}\sum_{n=1}^{N} P_{nm}\frac{\partial A_n}{\partial x} = 0, \tag{2}$$

where $P_{nm} = \delta_{nm} - \frac{q_{nm}}{J_2^2(j_{1,m})}$, $\delta_{nm} = \begin{cases}1, n = m\\ 0, n \neq m\end{cases}$. For $N = 1$ in (2), the last term can be excluded by Galilean transformation and hence, we consider (2) in the simplest non-trivial case $N = 2$. Let in (2), for $N = 2$, amplitudes A_1 and A_2 are: $A_1 = \sum_{n=1}^{M} A_{n1}e^{ix2\pi\alpha n}$, $A_2 = \sum_{n=1}^{M} A_{n2}e^{ix2\pi\beta n}$, A_{n1}, A_{n2} are constants. A complementary to the Reynolds number Re parameter can be defined as $p = \frac{\alpha}{\beta}$. From (2) for N=2, we get by Bubnov-Galerkin weighed differences method a system from which the condition of linear exponential instability with $\lambda_1 > 0$ in (2), is got for $\mathrm{Re} >> 1$, as:

$$\mathrm{Re} > \mathrm{Re}_{th} = \pi^2(1-p)^2\sqrt{F}\,/\,P_{12}P_{21}p^2|S|, \tag{3}$$

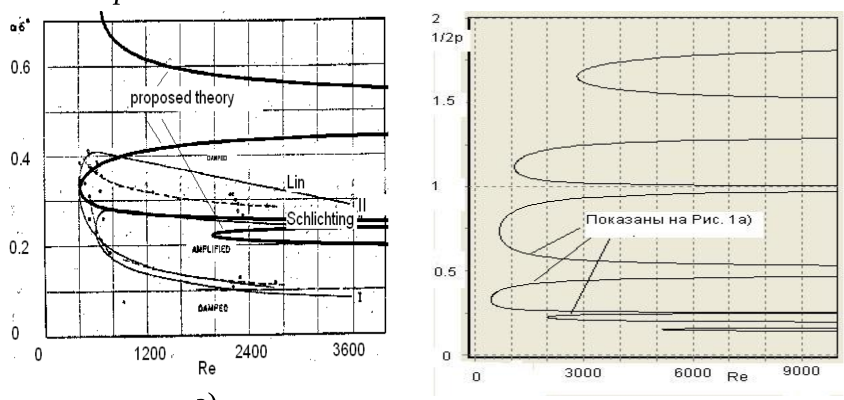
where $S = \sin\pi p\sin(\pi/p)\sin\pi(p + 1/p)$, $B = \frac{S}{|S|}(pP_{11} - P_{22})$,

$$F = (\gamma_{1,2}^2 + \gamma_{1,1}^2)(1+p^2)A^2 + (\gamma_{1,2}^2 - \gamma_{1,1}^2)(1-p^2)B^2 +$$

$$2AB(\gamma_{1,2}^2 - p^2\gamma_{1,1}^2),\ A^2 = B^2 - \frac{4S\,P_{12}P_{21}p^2 ctg\,\pi(p + \frac{1}{p})}{\pi^2(1-p)^2},$$ for

P_{11}, P_{22}, P_{12}, P_{21} from (3) , because for any p it's take place inequality

$A^2>0$. The value of the absolute minimum $\tilde{\mathrm{Re}}_{th}^{\min} \approx 442$ in (3) is reached for $p \approx 1.53..$, and close to it $\tilde{\mathrm{Re}}_{th}^{\min} \approx 448$ takes place for the same p.



a) b)

Fig. 1. Family of the six curves of neutral stability (with $\lambda_1 = 0$), according to (3) . On Fig..1a), scaled plots of three of them are given (they are noted also on Fig. 1b)). Meanwhile, the mean, second from below, instability region is bounded by the curve corresponding to the value β_0=0.463 (for p=1.527), and the lower one to β_0=1.099 (for p=2.239). On Fig.1a), we give overlapping with a figure from [23] (see Fig.12 in [23]) under condition that formally, $1/2p=\alpha\delta^*$. In [23], α is the wave disturbance number, and δ^* is the boundary layer shift thickness when streamlining a thin plate. On Fig.1a, points and dashes correspond to the experiment [23], and the solid lines correspond to the Shlihting theory (the lower, denoted by I) and of Lin (the upper, denoted by II).

4. Discussion and conclusion

Found value $\tilde{\mathrm{Re}}_{th}^{\min} \approx 448$ corresponds to the interval of values $\mathrm{Re} \in 300 \div 500$, noted in experimental observation of the threshold transition of the laminar resistance law (for a flow in the pipe) to another already non laminar (but yet not obviously turbulent) resistance mode [2, 20] and for TS waves in the near wall region of the boundary layer [19]. Observed [1,21,22] sensitivity of Re_{th} to the initial disturbances dependency (3) on p : Re_{th} in (3) changes nearly 600 times when p changes from 0,12 to 0,11. In the scaled form, fragments of the neutral curve, corresponding to the condition (3) (see Fig. 1b), are given on Fig. 1a) in the form of dependency of the value *1/2p* on *Re*. They are plotted on the taken from [23] figure (see Fig.12 in [23]), on which theoretical (Lin, Shlihting) neutral curves and respective experimental data defining instability threshold in a boundary layer, are given. Fig. 1a) allows concluding on similarity of the linear dissipative instability mechanisms for the HP flow and TS waves excitation. Thus, the suggested theory agrees with the observation data and results of the non-linear theory. We are grateful to S.I. Anisimov, G.S. Golitsin, V. P. Goncharov, E.A. Novikov, and N.A. Inogamov for useful comments and interest to the work.

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