NONLINEAR DYNAMICS OF MAGNETOHYDRODYNAMIC SHALLOW WATER FLOWS OVER AN ARBITRARY SURFACE

K. V. Karelsky [1], A.S. Petrosyan [1,2]

[1] Space Research Institute of the Russian Academy of Sciences
[2] Moscow Institute of Physics and Technology, (State University)

Abstract. The magnetohydrodynamic equations system for heavy fluid over an arbitrary surface in shallow water approximation is studied. All self-similar discontinuous and continuous solutions are found. The exact explicit solutions of initial discontinuity decay problem over a flat plane and a slope are found. It is shown that the initial discontinuity decay solution is represented by one of five possible wave configurations. For each configuration the necessary and sufficient conditions for its realization are found. The change of dependent and independent variables transforming the initial equations over a slope to those over a flat plane is found.

1. Introduction

The shallow water magnetohydrodynamic (SMHD) equations are the alternative to solving of full set of magnetohydrodynamic equations for a heavy fluid with a free surface. These equations are derived from the magnetohydrodynamic equations for an incompressible nonviscous fluid layer in the gravity field assuming the pressure is hydrostatic, using the depth averaging, and considering that the fluid layer depth is much smaller than the problem characteristic size. The derived system (Gilman, 2000; De Sterck, 2001) is important in many applications of magnetohydrodynamics to astrophysical and engineering problems. The magnetohydrodynamic shallow water approximation is widely used for the solar tachocline study (Gilman, 2000; Miesch and Gilman, 2004; Zaqarashvili et al, 2009; Zaqarashvili et al, 2010), for the description of spread of matter over a neutron-star surface during disc accretion (Inogamov and Sunyaev, 1999; Inogamov and Sunyaev, 2010), for the study of neutron-star atmosphere dynamics (Spitkovsky et al, 2002; Heng and Spitkovsky, 2009), for the study of extrasolar planets (Cho, 2008).

In the present work simple wave solutions for the shallow water magnetohydrodynamic (SMHD) equations over a non-flat plane are studied. It is shown these solutions exist only for the class of underlying surfaces that are slopes of constant inclination. Magnetogravity rarefaction wave, magnetogravity shock wave and Alfvenic wave solutions for slopes are found. Characteristics of these waves are parabolas transforming to straight lines in case of flat plane. These particular waves are fundamental for studying of nonlinear wave phenomena over a non-flat surface. The change of dependent and independent variables transforming the SMHD equations on a slope to those on a flat plane is found through analyzing the obtained solutions. It is used to find the exact solution of the initial discontinuity decay problem for SMHD equation system over a slope.

2. Initial equations. Simple waves

The magnetohydrodynamic shallow water equations over an arbitrary boundary are obtained from the classical magnetohydrodynamic equations written for the fluid layer with a free surface in the gravity field over an arbitrary boundary. There the Z axis is parallel to the gravity force vector and is opposite in direction to that of gravity force. Assuming the layer depth is small compared to the characteristic size of the studied phenomena and the full pressure (the sum of magnetic and hydrodynamic pressures) is hydrostatic the mentioned system is averaged over a fluid layer depth neglecting the squares of velocity and magnetic field deviations.

This system was first obtained by Gilman (2000) for flat plane. In one-dimensional case for arbitrary bed it takes the following form:

$$\frac{\partial h}{\partial t}+\frac{\partial hu_1}{\partial x}=0 \tag{1}$$

$$\frac{\partial hu_1}{\partial t}+\frac{\partial\left(hu_1^2-hB_1^2+gh^2/2\right)}{\partial x}=-gh\frac{\partial b}{\partial x} \tag{2}$$

$$\frac{\partial hu_2}{\partial t}+\frac{\partial\left(hu_1u_2-hB_1B_2\right)}{\partial x}=0 \tag{3}$$

$$\frac{\partial hB_1}{\partial t}=0 \tag{4}$$

$$\frac{\partial hB_2}{\partial t}+\frac{\partial\left(hB_2u_1-hB_1u_2\right)}{\partial x}=0 \tag{5}$$

$$\frac{\partial hB_1}{\partial x} = 0 \tag{6}$$

There x and t are the spatial and temporal coordinates correspondingly, $h(x,t)$ is the fluid depth, $u_1(x,t)$ and $u_2(x,t)$ are fluid velocities along X and Y axes respectively, $B_1(x,t)$ and $B_2(x,t)$ are magnetic field components along X and Y axes respectively and g is the gravitational constant.

It immediately follows from equations (1.4, 2) that

$$hB_1 = const \tag{7}$$

Thus we use (7) instead of (4) later.

According to the hyperbolic equations theory a Riemann wave is defined as the solution in which all but one Riemann invariants remain constant. However, classical Riemann wave solutions do not satisfy Riemann invariant form of equations due to the presence of function $-g\partial b/\partial x$ in the right-hand side of the equations. The Riemann wave solution thus defined as a solution satisfying all but one equations of mentioned equations. It is shown these solutions can exist only exist only for underlying surface $b(x)$ determined by $\partial^2 / \partial x^2 (b) \equiv 0$ equation, i.e. $b = kx + b_0$.

For slopes, two types of centered simple waves are found: magnetogravity rarefaction wave turned back and magnetogravity rarefaction wave turned forward. For a magnetogravity Riemann wave turned back the following relations

$$\begin{aligned} &B_1(x,t)h(x,t) = B_1(x_0,0)h(x_0,0) \\ &B_2(x,t) = B_2(x_0,0) \\ &u_2(x,t) = u_2(x_0,0) \\ &u_1(x,t) + \varphi(x,t) + gkt = u_1(x_0,0) + \varphi(x_0,0) \end{aligned} \tag{8}$$

are satisfied in the domain of the wave. Moreover, along the lines

$$\frac{dx}{dt} = u_1(x_0,0) - c_g(x_0,0) - gkt \tag{9}$$

the equation

$$u_1(x,t) - \varphi(x,t) + gkt = u_1(x_0,0) - \varphi(x_0,0) \tag{10}$$

is satisfied as well.

For a magnetogravity Riemann wave turned forward the following relations

$$\begin{aligned} &B_1(x,t)h(x,t)=B_1(x_0,0)h(x_0,0) \\ &B_2(x,t)=B_2(x_0,0) \\ &u_2(x,t)=u_2(x_0,0) \\ &u_1(x,t)-\varphi(x,t)+gkt=u_1(x_0,0)-\varphi(x_0,0) \end{aligned} \tag{11}$$

are satisfied in the domain of the wave. Moreover, along the lines

$$\frac{dx}{dt}=u_1(x_0,0)+c_g(x_0,0)-gkt \tag{12}$$

the equation

$$u_1(x,t)+\varphi(x,t)+gkt=u_1(x_0,0)+\varphi(x_0,0) \tag{13}$$

is satisfied as well.

Discontinuous solutions of two types were also found: magnetogravity shock waves and Alfwenic waves. For magnetogravity shock waves the following relations hold:

$$\begin{gathered} h_I B_I = h_{II} B_{II} \\ D=\frac{h_I u_{1I}-h_{II}u_{1II}}{h_I-h_{II}} \\ u_{1I}-u_{1II}=\pm(h_I-h_{II})\sqrt{\frac{g/2(h_I+h_{II})+(B_{1I}h_I)^2/(h_I h_{II})}{h_I h_{II}}} \end{gathered} \tag{14}$$

For Alfwenic waves the following relations hold:

$$\begin{aligned} &D=u_1-B_1\frac{B_{2I}-B_{2II}}{u_{2I}-u_{2II}} \\ &(B_{2I}-B_{2II})^2=(u_{2I}-u_{2II})^2 \end{aligned} \tag{15}$$

We also found the change of variables, transforming SMHD equations over a slope to ones over a flat plane:

$$\begin{aligned} &\tilde{x}\to x+gkt^2/2 \\ &\tilde{t}\to t \\ &\tilde{u}_1=u_1+gkt \end{aligned} \tag{16}$$

It follows from (21) that

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial \tilde{t}}+gk\tilde{t}\,\frac{\partial}{\partial \tilde{x}}$$
$$\frac{\partial}{\partial x}=\frac{\partial}{\partial \tilde{x}} \tag{17}$$

After using the transformation (22) the initial system becomes the magnetohydrodynamic shallow water equation system on a flat surface (with $k=0$). This transformation is used below to solve the initial discontinuity decay problem over a slope reducing it to the discontinuity decay problem over a flat plane solved by Karelsky et al (2010).

4. Conclusions

In the present work the nonlinear dynamics of shallow water magnetohydrodynamic flows of heavy fluid is studied. It is shown simple wave solutions exist only when the surface is slope. All simple wave solutions for slopes are found: Alfvenic waves and magnetogravity waves. The change of variables transforming SMHD equations over a slope to ones over a flat plane is found. The exact explicit solution of initial discontinuity decay problem over a slope is found. It is shown these solutions are represented by one of the five wave configurations.

It follows from the obtained results that the solution of initial discontinuity decay is the superposition of two solutions: the initial discontinuity decay solution for shallow water without magnetic field (with modified sound velocity $c_g=\sqrt{B_1^2+gh}$) and two Alfwenic waves. When $B_1\equiv 0$ two Alfwenic waves merge and become the contact discontinuity. The 'two hydrodynamic rarefaction waves and a vacuum region between them' configuration differs from other configurations and can be realized only when normal component of magnetic field equals to zero.

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