ERGODIC DISTRIBUTION OF CHARGED PARTICLES IN COULOMB FIELD

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Abstract. Finite motion of electrons after instantaneous switching on of an external positive point-like charge is considered. The trapped particle distribution function averaged over periods of the motion is determined. Contribution of these electrons to the total perturbation of plasma density is calculated. It is shown that the trapped particle contribution dominates at small distances from the charge, whereas it is negligible at large distances.

1. Introduction

Shielding of a charge in a plasma is usually described by Debye formula. As is known, the derivation of the Debye expression is based on linear approximation. A possible formulation of the problem presumes instantaneous appearance of an external point-like charge in a uniform isotropic plasma [1]. At long times, solution of the problem yields the Debye formula for the electrostatic potential. However, according to this formula, intensity of the electric field takes very large values in the vicinity of the charge thereby contradicting to applicability of the perturbation theory. Another possible way to calculate the spatial dependence of the self-consistent electrostatic potential consists in a nonlinear analysis of equilibrium states of the plasma, for example, similarly to calculations carried out in [2]. However, in doing so, distribution function of particles moving on finite orbits remains undetermined. Thus, the description of the nonlinear shielding in collisionless plasmas is an open question up to now.

In this paper, a method for computing the trapped particle distribution within the framework of a nonlinear approach to the initial value problem [1] is suggested. Although such a rigorous analysis of the problem becomes too difficult, it is still possible to find some averaged asymptotic trapped particle distribution at long times. The technique in use is similar to the method utilized in the work [3], wherein averaged, so called "ergodic", distribution function has been found in application to collisionless damping of a finite amplitude wave. The main idea of the method is that the motion of charged particles may be considered in a given field of the wave. Thereafter, their averaged distribution function is found and used in computations of the moments of the distribution function appearing in the equations for the electric field. Below, it is shown that such an approach is also well justified as applied to studies of the shielding phenomena in collisionless plasmas.

2. Formulation of the problem and basic equations

Similarly to the problem formulated in [1], let us assume that an external point-like charge appears in a plasma at the instant of time t=0. After relaxation of transient wave processes, the state of the plasma tends to some asymptotic equilibrium. It is rather difficult to describe entirely the plasma dynamics at all times. Because of this, we will be interested mainly in this steady state. The manner to find distribution function of free (passing) particles moving infinitely is well known [2]. However, it is not easy to determine the trapped particle distribution.

At small distances from the charge, according to the Debye expression, the electrostatic potential is approximated by the Coulomb formula $\phi = Q/r$. Assuming that nonlinear shielding, as well as the linear one, does not lead to any significant deviation from Coulomb's law at these distances, let us presume that the field remains Coulomb's at any instant of time. As we will see later, this assumption is really justified, since corrections to the Coulomb's expression caused by weak shielding of the charge must be small at the distances less or of the order of Debye length. Moreover, as is shown below, the trapped particles contribute substantially to plasma density perturbation just at small distances from the charge. Thus, the problem is reduced to the calculation of the trapped particle distribution in the Coulomb field.

Here, for definiteness, let us consider the shielding of a positive point-like charge Q by plasma electrons under assumption of immobile ions. Assuming the instantaneous appearance of the external charge, we will calculate the trapped electron distribution function. For brevity, the following units of measurement are utilized

$$[t] = \omega_p^{-1} \quad , \quad [v] = c, \quad [r] = d \equiv c/\omega_p, \quad [n] = n_0 \quad , \quad [\phi] = mc^2/e \quad ,$$

$$[W] = mc^2 \quad , \quad [Q] = 4\pi e n_0 d^3 \quad , \quad [f] = n_0/c^3 \quad , \quad [M] = mcd \quad ,$$

where the conventional notation is in use, except for c, i.e. n_0 is the unperturbed plasma number density, $\omega_p = (4\pi e^2 n_0/m)^{1/2}$ is the electron plasma frequency, c is a typical value of electron velocity, the choice of which can be specified in correspondence with a form of unperturbed distribution function f_0, d is the effective Debye length, ϕ is the electrostatic potential, W is the energy of electron, and M is its angular momentum.

The motion of a particle in a central field is well studied (e.g. in [2]). It is important that corresponding equations of motion can be conveniently reduced to the following equations describing the motion in radial direction with one degree of freedom

$$\frac{dr}{dt} = v_r \quad , \quad \frac{dv_r}{dt} = -\frac{\partial U}{\partial r} = \frac{M^2}{r^3} - \frac{Q}{r^2} \quad ,$$

where v_r is radial velocity, and

$$U = \frac{M^2}{2r^2} - \frac{Q}{r}$$

is the effective potential taking into account the action of centrifugal force. Corresponding Vlasov equation reads

$$\frac{\partial f}{\partial t} + v_r \frac{\partial f}{\partial r} - \frac{\partial U}{\partial r}\frac{\partial f}{\partial v_r} = 0 \quad .$$

As an initial condition, any isotropic in velocity space distribution function f_0 (u), u = |v| can be given, for example, mono-energetic distribution $f_0 = (1/4\pi)\delta(u-1)$ or Maxwellian one.

3. Ergodic trapped electron distribution function

If necessary, the Vlasov equation can be solved rigorously. Physical interpretation of the solution is as follows. At t=0, the energy of an electron decreases sharply by the value $\phi(r_0)$, where $r_0 = r(0)$ is the initial position of the electron, i.e. $W = \varepsilon - \phi(r_0)$, at t>0, where ε is the initial kinetic energy of the particle. The energy of the trapped electrons becomes negative W<0. As the frequency of electron oscillations in the effective potential well depends on the energy, a phase mixing process takes place (for more detail see, e.g. [3]). As time is going, the trapped particle distribution function becomes strongly oscillating in phase space. However, averaged over time, so called “ergodic”, distribution does not depend on time. The ergodic distribution function can be found by means of the following procedure of averaging

$$< f >= (2/T) \int_{r_-}^{r_+} \frac{dr f_0}{v_r(r)} = \int_{r_-}^{r_+} \frac{dr f_0}{v_r(r)} \left(\int_{r_-}^{r_+} \frac{dr}{v_r(r)} \right)^{-1} \quad .$$

In essence, this procedure is similar to the averaging carried out in [3]. The distinctions are caused by the different spatial dependencies of the electrostatic potential of the plane wave [3] and the effective potential under consideration. In addition, the effective potential depends on the parameter M. Taking into account the interrelation between the energy of a particle before, W= $\mathcal{E}$, and after the appearance of the external charge $\mathcal{E}$ = W + Q/r = Q/r - w, (w = -W > 0), and changing the integration variable r for ☜, we come to the following result

$$< f >= \frac{Q}{\pi M}(2w)^{3/2} \int_{\mathcal{E}_-}^{\mathcal{E}_+} \frac{d\mathcal{E} f_0(\mathcal{E})}{(\mathcal{E}+w)^2\sqrt{(\mathcal{E}-\mathcal{E}_-)(\mathcal{E}_+-\mathcal{E})}} \quad ,$$

where f_0 ($\mathcal{E}$) is the unperturbed distribution function, and $\mathcal{E}_\pm = Q/r_\pm - w$ are, respectively, the maximum and minimum initial kinetic energies of the trapped electrons at turning points of trajectory for a given value of M. The physical meaning of the averaging may be interpreted as follows. Every electron is moving along a spatially limited trajectory. In some sense, the averaging “washes out” the individual particle along the trajectory in accordance with relative residence time in every point of the trajectory.

4. Electron number density

Going to the integration variables W and M, it is not difficult to express the electron density n in the form (see e.g. [2] for detail)

$$\frac{n}{4\pi} = \int_0^\infty \frac{dM M}{r^2} \int_U^\infty \frac{dW f(M,W)}{\sqrt{2(W-U)}} = \int_0^\infty d\mathcal{E}_\perp \int_{\mathcal{E}_\perp - \phi}^\infty \frac{dW f(\mathcal{E}_\perp, W)}{\sqrt{2(W+\phi-\mathcal{E}_\perp)}} \, ,$$

where $\mathcal{E}_\perp = v_\perp^2/2 = M^2/2r^2$. At $t \to \infty$, the steady state distribution function of free particles is determined by the boundary condition at infinity $f(r \to \infty) = f_0$, so that the density of the free particles can be easily found similarly to the calculations carried out in [2]. In particular, in the case of the mono-energetic f_0 , taking into account the additional restriction for the free electrons W>0, it is easy to obtain the expression for the perturbation of the free electron density

$$\tilde{n}_P = \sqrt{1+2\phi} - 1 \quad .$$

For the trapped electrons, one need to take into account the additional restriction W < 0 , (w = -W > 0) and the limitations connected with the positiveness of the radicand in the denominator of the expression for ergodic trapped electron distribution. Then, the integration leads to the result

$$n_T(\phi) = 2(2\phi)^{3/2} \int_0^\infty d\mathcal{E} f_0(\phi\mathcal{E}) I(\sqrt{\mathcal{E}}) \quad ,$$

where the function I=I(a) is defined by

$$I(a) \equiv \int_0^1 \frac{dx x(1-x^2)^{1/2}}{(a^2+1-x^2)^3} \ln\left[\frac{a+x(a^2+1-x^2)}{a-x(a^2+1-x^2)}\right]^2 \quad .$$

These expressions determine dependence of the trapped particle density on the value of the electrostatic potential as well as on the specific form of the undisturbed distribution function. Integration in the last formula leads to a rather awkward expression. However, it is not hard to examine the behavior of n_T at small and large ϕ. The calculation is particular simple in the case of mono-energetic distribution

$$n_T \simeq (3/4)(2\phi)^4 \quad \text{at} \quad \phi \ll 1 \quad , \quad \text{and} \quad n_T \simeq 2(2\phi)^{3/4} \quad \text{at} \quad \phi \gg 1 \, .$$

Therefore, the trapped electron density is very small far from the external charge, while in the vicinity of the charge it takes large values. The corresponding consequences are discussed in the next section.

5. Trapped electron contribution to plasma density perturbation

Here, we will not be concerned with spatial dependence of the self-consistent electrostatic potential described by a solution of the corresponding Poisson equation. Nevertheless, it is of interest to compare the contributions of the trapped and free electron populations to the total plasma density perturbation, i.e. to the right-hand side of the Poisson equation

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{d\phi}{dr} + Q\delta(r) = \tilde{n} = \tilde{n}_P(\phi) + n_T(\phi) = \sqrt{1+2\phi} - 1 + n_T(\phi) \quad .$$

At large distances from the external charge (ϕ << 1), neglecting the small n_T and expanding the square root, we come to linear equation. Its solution is given by Debye formula. However, at small distances

($\phi >> 1$), the linearization is not justified. Moreover, the trapped electron contribution proportional to $\phi^{3/4}$ exceeds the free electron density perturbation. Since both of the terms are less than ϕ , the nonlinear shielding of the external charge must be weaker than the Debye shielding.

6. Discussion and conclusions

A way to determine contribution of the trapped particles to the perturbation of the plasma density has been suggested. The trapped electron density has been calculated on the basis of the concept of ergodic distribution arising in the course of evolution of the trapped particle distribution in the Coulomb field. It can be shown that, at these distances, the dependence $n_T \propto \phi^{3/4}$ takes place also in the case of Maxwellian distribution f_0, and hence, this scaling is quite general in nature. Since the nonlinear plasma response is less than the linear analogue, it is natural to expect a certain displacement of the region of the Debye's exponential decrease in the electric field intensity toward larger radial distances. The developed approach and found expressions for the trapped particle number density may be applied to studies of nonlinear shielding of small charged bodies in collisionless plasmas.

References

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